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Abstract. The BRST operator defining the cohomology groups for the quantum SU(2) group is constructed.

1. Introduction

The non-commutative geometry [1] is now supposed to be of some relevance to physics. This is at least the case as far as quantum groups are concerned [2–5]. They are finding application in 2D solvable model S-matrices [6, 7], 3D Chern–Simmons theories [8] as well as conformally invariant theories in two dimensions [9]. There are also some attempts to find phenomenological applications [10].

The theory of quantum groups is quickly developing. The notions of quantum spaces (quantum hyperplane [11], quantum sphere [12]), differential calculi [5, 13, 14] and linear representations [3, 4] were introduced and studied in some detail. The aim of such investigations is to supply us, by generalizing the rich mathematical structures related to ordinary Lie groups, with suitable tools to analyse the problem of whether quantum groups provide the proper extension of the notion of symmetry in physics.

One of the most important notions in physics is that of gauge symmetry. Gauge theories are most conveniently quantized using the BRST method. It is well known [15] that BRST symmetry is closely related to the cohomology theory on gauge groups. In particular, the standard cohomology on compact Lie groups can be viewed as a BRST transformation for constant field configurations [16].

If one believes seriously that the notion of quantum groups provides a proper generalization of ordinary symmetry one can pose the question of what is the generalization of gauge or BRST symmetry. We propose to use the observation made in [16] as a starting point in answering this question. Once the differential calculus on the compact quantum group has been introduced† one can study the relevant cohomology theory [3]. We follow the lines of [16] and construct the 'BRST operator' realizing the same cohomology theory. This operator is constructed in terms of 'ghost' variables belonging to the deformed Grassman algebra and the linear representation of deformed Lie algebra. We study here only the simplest example—the twisted SU(2) group [3]. The general construction for bicovariant differential calculi [5, 13] is postponed to a further publication [17].

† We use the formalism developed by Woronowicz [3–5].

2. Classical cohomology

It is well known [18] that for the compact Lie groups De Rham cohomology (and also the equivalent singular cohomology) can be reduced to the one based on left- or right-invariant differential forms. This can be understood as follows. One first notes that any form can be written in a basis spanned by (say) left-invariant forms. Due to the identity

$$\mathrm{d}f \equiv \mathrm{d}x^{\alpha}\,\partial_{\alpha} f = \omega^{\alpha}\nabla_{\alpha} f \tag{1}$$

where ω^{α} are the left-invariant forms and ∇_{α} the left-invariant fields, one can write

$$\begin{aligned}\mathrm{d}(f_{\alpha_1\ldots\alpha_k}\omega^{\alpha_n}\wedge\ldots\wedge\omega^{\alpha_k})\\ =\nabla_{\alpha} f_{\alpha_1\ldots\alpha_k}\omega^{\alpha}\wedge\omega^{\alpha_1}\wedge\ldots\wedge\omega^{\alpha_k}+f_{\alpha_1\ldots\alpha_k}\mathrm{d}(\omega^{\alpha_1}\wedge\ldots\wedge\omega^{\alpha_k}).\end{aligned} \tag{2}$$

Using the Peter–Weyl theorem we next observe that $f_{\alpha_1\ldots\alpha_k}$ can be replaced by the matrix elements of irreducible representations. Then the action of ∇_{α} in (2) is equivalent to an infinitesimal group transformation. If we note that (i) the infinitesimal group transformation of the matrix elements of a given representation is given in terms of the representation of the corresponding Lie algebra and (ii) $\mathrm{d}\omega^{\alpha}$ is determined by Cartan–Maurer equations, we conclude that our cohomology theory is reduced to the following algebraic problem [19]. We consider the tensor product $V\otimes\Lambda(\omega)$ where V is the representation space of the Lie algebra under consideration and $\Lambda(\omega)$ is the exterior algebra spanned by ω's, and define the operator

$$\mathrm{d}(f\otimes\zeta)=G_{\alpha} f\otimes\omega^{\alpha}\wedge\zeta+f\otimes\mathrm{d}\zeta \qquad f\in V \qquad \zeta\in\Lambda(\omega) \qquad G_{\alpha}\in\mathrm{End}(V). \tag{3}$$

Here G_{α} are the generators of the Lie algebra in a given representation and $\mathrm{d}\omega^{\alpha}$ is determined by Cartan–Maurer equations. The cohomology groups of the operator (3) determine those of the group manifold. It is not difficult to show that for compact Lie groups (algebras) the non-trivial cohomology groups are obtained only provided the representation $\{G_{\alpha}\}$ contains the trivial subrepresentation. If we finally note that the regular representation contains the trivial one exactly once we conclude that De Rham cohomology is equivalent to the one based on invariant forms.

The cohomology defined by (3) can be realized as BRST transformations for constant field configurations [16]. To this end one introduces the ghost variables and ghost momenta c^{α} and Π_{α}, which fulfil the (anti)commutation rules

$$\{c^{\alpha},c^{\beta}\}=0 \qquad \{\Pi_{\alpha},\Pi_{\beta}\}=0 \qquad \{c^{\alpha},\Pi_{\beta}\}=\delta^{\alpha}{}_{\beta} \tag{4}$$

and define the BRST operator

$$\Omega=c^{\alpha}G_{\alpha}+\frac{\mathrm{i}}{2}c^{\gamma}c^{\beta}f^{\alpha}{}_{\beta\gamma}\Pi_{\alpha}. \tag{5}$$

Here $f^{\alpha}{}_{\beta\gamma}$ are the Lie algebra structure constants, $[G_{\alpha},G_{\beta}]=\mathrm{i}f_{\alpha\beta}{}^{\gamma}G_{\gamma}$. Ω is nilpotent and defines the same cohomology as that given by (3). This is easily seen by noting that the representation space for the operators G_{α}, c^{α} and Π_{α} consists of vectors of the form [16]

$$\begin{aligned}&\psi[c]=\sum_{k=0}^{n}\frac{1}{k!}c^{\alpha_1}\ldots c^{\alpha_k}\psi^{(k)}_{[\alpha_1\ldots\alpha_k]}\\ &\psi^{(k)}_{[\alpha_1\ldots\alpha_k]}\in V \qquad \Pi_{\alpha}\psi^{(k)}_{[\alpha_1\ldots\alpha_k]}=0.\end{aligned} \tag{6}$$

One can also define the anti-BRST operator Ω^+ [16], which is the Hermitian conjugate of Ω with respect to a positive-definite scalar product, and the analogue of the Laplacian

$$W=(\Omega+\Omega^+)^2=\Omega\Omega^+ +\Omega^+\Omega. \tag{7}$$

As an easy consequence of representation theory for BRST algebra one obtains the 'Hodge' decomposition theorem:

$$\psi=w+\Omega\chi+\Omega^+\varphi \qquad Ww=0. \tag{8}$$

Therefore the cohomology classes are determined by 'harmonic' states w. Now, W can be written as a sum of squares, one of which is simply the Casimir operator for the group under consideration. Consequently, the harmonic states must be singlets of Lie algebra. This is equivalent to the statement given previously: in order to obtain non-trivial cohomology the representation under consideration must contain a trivial subrepresentation. Let us stress again that if we choose the generators G_α belonging to the regular representation then (i) the operators d in (1) and (3) are equivalent, (ii) the action of Ω (Ω^+) becomes equivalent to that of $\mathrm{d}(\delta)$ and (iii) (8) becomes the Hodge decomposition theorem.

3. Quantum cohomology

In order to carry over the above construction to the case of the twisted SU(2) group let us recall the structure of differential calculus over $SU_\mu(2)$. The Cartan forms ω^0, ω^1, ω^2 fulfil the commutation rules

$$\begin{aligned}
&\omega^0\wedge\omega^0=\omega^1\wedge\omega^1=\omega^2\wedge\omega^2=0\\
&\omega^1\wedge\omega^0=-\mu^4\omega^0\wedge\omega^1\\
&\omega^2\wedge\omega^0=-\mu^2\omega^0\wedge\omega^2\\
&\omega^2\wedge\omega^1=-\mu^4\omega^1\wedge\omega^2
\end{aligned} \tag{9}$$

and Cartan–Maurer equations

$$\begin{aligned}
&\mathrm{d}\omega^0=\mu^2(1+\mu^2)\omega^0\wedge\omega^1\\
&\mathrm{d}\omega^1=\mu\omega^0\wedge\omega^2\\
&\mathrm{d}\omega^2=\mu^2(1+\mu^2)\omega^1\wedge\omega^2.
\end{aligned} \tag{10}$$

The Lie algebra of $SU_\mu(2)$ reads [3]

$$\begin{aligned}
&\mu G_2G_0-\frac{1}{\mu}G_0G_2=G_1\\
&\mu^2G_1G_0-\frac{1}{\mu^2}G_0G_1=(1+\mu^2)G_0\\
&\mu^2G_2G_1-\frac{1}{\mu^2}G_1G_2=(1+\mu^2)G_2.
\end{aligned} \tag{11}$$

In order to construct the BRST operator we first introduce three ghost variables fulfilling the same commutation rules as ωs:

$$\begin{aligned} c^1c^0 &= -\mu^4 c^0 c^1 \\ c^2c^0 &= -\mu^2 c^0 c^2 \\ c^2c^1 &= -\mu^4 c^1 c^2 \\ (c^0)^2 &= (c^1)^2 = (c^2)^2 = 0. \end{aligned} \tag{12}$$

We have to also introduce the conjugate momenta Π_0, Π_1, Π_2. The commutation rules for them which are consistent with the associativity read

$$\begin{aligned} &\Pi_0^2 = \Pi_1^2 = \Pi_2^2 = 0 && \{c^0, \Pi_0\} = \{c^1, \Pi_1\} = \{c^2, \Pi_2\} = 1 \\ &\Pi_1\Pi_0 = -\mu^4\Pi_0\Pi_1 && \Pi_0 c^1 = -\mu^4 c^1 \Pi_0,\ c^0\Pi_1 = -\mu^4 \Pi_1 c^0 \\ &\Pi_2\Pi_0 = -\mu^2\Pi_0\Pi_2 && \Pi_0 c^2 = -\mu^2 c^2 \Pi_0,\ c^0\Pi_2 = -\mu^2 \Pi_2 c^0 \\ &\Pi_2\Pi_1 = -\mu^4\Pi_1\Pi_2 && \Pi_1 c^2 = -\mu^4 c^2 \Pi_1,\ c^1\Pi_2 = -\mu^4 \Pi_2 c^1. \end{aligned} \tag{13}$$

Let us now consider the following operator:

$$\begin{aligned} \Omega &= c^0 G_0 + c^1 G_1 + c^2 G_2 + \tilde{\Omega} \\ \tilde{\Omega} &= \mu^2(1+\mu^2)(c^0c^1\Pi_0 + c^1c^2\Pi_2) + \mu c^0 c^2 \Pi_1 + \mu^2(1+\mu^2)(\mu^4-1)c^0c^1c^2\Pi_2\Pi_0. \end{aligned} \tag{14}$$

Here G_α belong to some linear representation of the algebra (11). It is easy to check that Ω is nilpotent.

Woronowicz [3] defines the cohomology theory using the basic equations (3), (9), (10) and (11). In order to prove that our operator Ω gives the same cohomology classes it is sufficient to note that:

(i) The operators c^α, Π_α and G_α act in the space of vectors of the form

$$\psi[c] = \sum_{k=0}^{3} \sum_{\alpha_1<\ldots<\alpha_k} c^{\alpha_1}\ldots c^{\alpha_k}\psi^{(k)}_{\alpha_1\ldots\alpha_k} \qquad \Pi_\alpha \psi^{(k)}_{\alpha_1\ldots\alpha_k} = 0 \qquad \psi^{(k)}_{\alpha_1\ldots\alpha_k} \in V. \tag{15}$$

(ii) The following anticommutation rules hold:

$$\begin{aligned} \{\tilde{\Omega}, c^0\} &= \mu^2(1+\mu^2)c^0c^1 \\ \{\tilde{\Omega}, c^1\} &= \mu c^0 c^2 \\ \{\tilde{\Omega}, c^2\} &= \mu^2(1+\mu^2)c^1c^2. \end{aligned} \tag{16}$$

Therefore, with the definition adopted, Ω is a good candidate for a BRST operator. Following the lines of [16] we introduce the scalar product of vectors (15)

$$(\phi[c], \psi[c]) = \sum_{k=0}^{3} \sum_{\alpha_1<\ldots<\alpha_k} (\phi^{(k)}_{\alpha_1\ldots\alpha_k}, \psi^{(k)}_{\alpha_1\ldots\alpha_k})_V. \tag{17}$$

Then we have

$$\begin{aligned} &(c^0)^+ = \Pi_0 \\ &(c^1)^+ = \Pi_1[1+(\mu^8-1)c^0\Pi_0] \qquad \Pi_1^+ = c^1[1+(\mu^{-8}-1)c^0\Pi_0] \\ &(c^2)^+ = \Pi_2[1+(\mu^4-1)c^0\Pi_0][1+(\mu^8-1)c^1\Pi_1] \\ &\Pi_2^+ = c^2[1+(\mu^{-4}-1)c^0\Pi_0][1+(\mu^{-8}-1)c^1\Pi_1]. \end{aligned} \tag{18}$$

The anti-BRST operator Ω^+ reads

$$\begin{aligned}\Omega^+ &= -\Pi_0\mu^{-1}G_2+\Pi_1[1+(\mu^8-1)c^0\Pi_0]G_1\\ &\qquad -\Pi_2[1+(\mu^4-1)c^0\Pi_0][1+(\mu^8-1)c^1\Pi_1]\mu G_0+\tilde\Omega^+\\ \tilde\Omega^+ &= -\mu^6(1+\mu^2)\Pi_1c^0\Pi_0-\mu^3c^1\Pi_0\Pi_2\\ &\qquad +\mu^2(1+\mu^2)[1+(\mu^8-1)c^0\Pi_0]c^2\Pi_2\Pi_1+\mu^6(1+\mu^2)(1-\mu^4)c^0\Pi_0c^2\Pi_2\Pi_1.\end{aligned} \tag{19}$$

As in [16] we can prove the decomposition theorem (equation (8)). In fact, it relies only on the structure of BRST algebra which is here the same. It again follows that the cohomology classes are determined by harmonic forms. One can check that the non-trivial cohomologies result from trivial subrepresentations only in accordance with [3].

Let us conclude with a few remarks. We have constructed the BRST operator which gives the realization of the cohomology theory of [3] in terms of deformed Grassman algebra. However, contrary to the 'classical' case, there seems to be no natural choice of differential calculus over a quantum group. The resulting cohomology groups depend of course on the choice of calculus. Therefore, the BRST operator (quantum BRST symmetry?) is not uniquely determined by the structure of the quantum group.

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